# Dynamics on the Unit Disk: Short Geodesics and Simple Cycles

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## 1 Introduction

In this paper we show that rotation cycles on $S^1$ for a proper holomorphic map $f : \Delta \to \Delta$ share several of the analytic, geometric and topological features of simple closed geodesics on a compact hyperbolic surface.

**Dynamics on the unit disk.** Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For $d > 1$ let $\mathcal{B}_d \cong \Delta^{(d-1)}$ denote the space of all proper holomorphic maps $f : \Delta \to \Delta$ of the form

$$f(z) = z \prod_1^{d-1} \left( \frac{z - a_i}{1 - \overline{a}_i z} \right),$$

$|a_i| < 1$. Every degree $d$ holomorphic map $g : \Delta \to \Delta$ with a fixed point in the disk can be put into the form above, by normalizing so its fixed point is $z = 0$.

The maps $f \in \mathcal{B}_d$ have the property that $f|S^1$ is measure-preserving and $|f'| > 1$ on the circle. Moreover, there is a unique *marking* homeomorphism $\phi_f : S^1 \to S^1$ that varies continuously with $f$, conjugates $f$ to $p_d(z) = z^d$,

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and satisfies $\phi_f(z) = z$ when $f = p_d$. We define the *length* on $f$ of a periodic cycle $C$ for $p_d$ by

$$L(C, f) = \log |(f^q)'(z)|, \tag{1.1}$$

where $q = |C|$ and $\phi_f(z) \in C$.

The *degree* of a cycle $C$ is the least $e > 0$ such that $p_d|C$ extends to a covering map of the circle of degree $e$. We say $C$ is *simple* if $\deg(p_d|C) = 1$; equivalently, if $p_d|C$ preserves its cyclic ordering. A finite collection of cycles $C_i$ is *binding* if $\deg(\bigcup C_i) = d$ and if $\bigcup C_i$ is not renormalizable (§7).

In this paper we establish four main results.

**Theorem 1.1** *Any cycle with $L(C, f) < \log 2$ is simple. All such cycles $C_i$ have the same rotation number, and $p_d|\bigcup C_i$ preserves the cyclic ordering of $\bigcup C_i$.*

**Theorem 1.2** *Every $f \in \mathcal{B}_d$ has a simple cycle $C$ with $L(C, f) = O(d)$.*

**Theorem 1.3** *Let $(C_i)_1^n$ be a binding collection of cycles. Then for any $M > 0$, the set of $f \in \mathcal{B}_d$ with $\sum_1^n L(C_i, f) \le M$ has compact closure in the moduli space of all rational maps of degree $d$.*

**Theorem 1.4** *The closure $E \subset S^1$ of the simple cycles for a given $f \in \mathcal{B}_d$ has Hausdorff dimension zero.*

See Theorems 4.1, 5.8, 7.1 and 2.2 below.

**Hyperbolic surfaces.** The results above echo the following fundamental facts about compact hyperbolic surfaces $X$ of genus $g > 1$:

1. The closed geodesics on $X$ of length less than $\log(3 + 2\sqrt{2})$ are simple and disjoint.
2. There exists a simple closed geodesic on $X$ with length $O(\log g)$.
3. If $(\gamma_i)_1^n$ is a binding collection of closed curves, then the locus in Teichmüller space $\mathcal{T}_g$ where $\sum L(\gamma_i, X) \le M$ is compact for any $M > 0$.[1]
4. The union of the simple geodesics on $X = \Delta/\Gamma$ is a closed set of Hausdorff dimension one.

See [Bus, §4, §5], [Ker, Lemma 3.1] and [BS] for proofs. Thus simple cycles behave in many ways like simple closed geodesics.

[1] A collection of closed curves is *binding* if their geodesic representatives cut $X$ into disks.

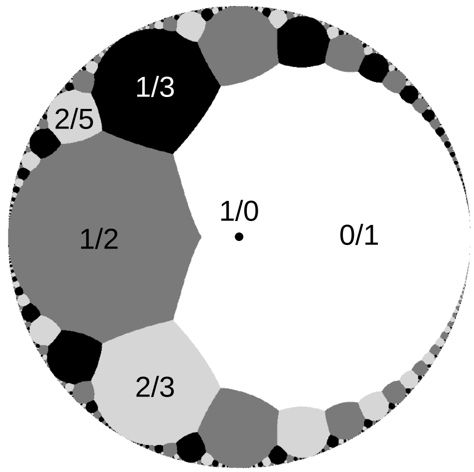


Figure 1. Tiling of $\Delta^*$ according to the slope of the shortest loop on the torus $\mathbb{C}^*/\alpha^{\mathbb{Z}}$.

**Rotation numbers and slopes.** Next we formulate a more direct connection between short cycles and short geodesics. Suppose $f \in \mathcal{B}_d$ satisfies $\alpha = f'(0) = \exp(2\pi i \tau) \neq 0$. The action of $\langle f \rangle$ on $\Delta$ (with the orbit of $z = 0$ removed) determines a natural *quotient torus*, isomorphic to

$$X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \cong \mathbb{C}^*/\alpha^{\mathbb{Z}}.$$

Let $L(p/q, X_\tau)$ denote the length of a closed geodesic on $X_\tau$ in the homotopy class $(-p, q)$, for the flat metric of area one. The slope $p/q \bmod 1$ which minimizes $L(p/q, X_\tau)$ depends only on $f'(0) \in \Delta^*$. The regions $T(p/q) \subset \Delta^*$ where a given slope is shortest rest on the corresponding roots of unity, and form a tiling of $\Delta^*$ (see Figure 1).

In §6 we will show:

**Theorem 1.5** *For any $f \in \mathcal{B}_d$ with $f'(0) \in T(p/q)$, there is a nonempty collection of compatible simple cycles $C_i$ with rotation number $p/q$ such that*

$$\frac{1}{L(p/q, X_\tau)^2} \leq \sum \frac{\pi}{L(C_i, f)} \leq \frac{1}{L(p/q, X_\tau)^2} + O(d),$$

*and all other cycles satisfy $L(C, f) > \epsilon_d > 0$.*

(Compatibility is defined in §2.) This result implies Theorem 1.2 and gives an alternate proof of Theorem 1.1 (with $\log 2$ replaced by $\epsilon_d$); it also yields:

**Corollary 1.6** *If a sequence $f_n \in \mathcal{B}_d$ satisfies $L(C, f_n) \to 0$, then $f_n'(0) \to \exp(2\pi i p/q)$ where $p/q$ is the rotation number of $C$.*

On the other hand, we will see in §3:

**Proposition 1.7** *If $f_n \in \mathcal{B}_d$ and $f_n'(0) \to \exp(2\pi i\theta)$ where $\theta$ is irrational, then $L(C, f_n) \to \infty$ for every cycle $C$.*

Thus the cycles of moderate length guaranteed by Theorem 1.2 may be forced to have very large periods.

**Petals.** The proof of Theorem 1.5 is illustrated in Figure 2. Consider a map $f \in \mathcal{B}_2$ with $f'(0) = \exp(2\pi i\tau) \in T(1/3)$, $\tau = 1/3 + i/10$. The dark petals shown in the figure form the preimage $\widetilde{A} \subset \Delta$ of an annulus $A$ in the homotopy class $[3\tau - 1]$ on the quotient torus for the attracting fixed point at $z = 0$. Any two adjacent rectangles within a petal give a fundamental domain for the action of $f$. The three largest petals join $z = 0$ to the repelling cycle on $S^1$ labeled by $C = (1/7, 2/7, 4/7)$. Thus a copy of $A$ embeds in the quotient for torus the repelling cycle as well; by the method of extremal length (§5), this gives an upper bound for $L(C, f)$ in terms of $L(1/3, X_\tau)$. (The lower bound comes from the holomorphic Lefschetz fixed-point theorem.)

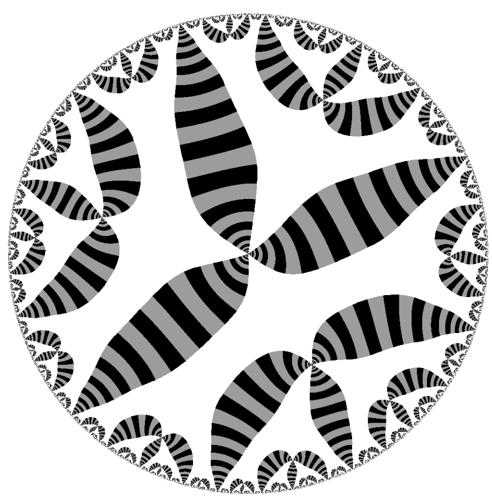

Figure 2. Petals joining $z = 0$ to the $(1, 2, 4)/7$ cycle on $S^1$.

**Rational maps.** Here is a related result from §5 for general rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Let $L(f) = \inf \log |\beta|$, where $\beta$ ranges over the multipliers of all repelling and indifferent periodic cycles for $f$.

**Theorem 1.8** *If $f_n \in \mathrm{Rat}_d$ and $L(f_n) \to \infty$, then the maps $f_n$ have fixed points $z_n$ with $f'(z_n) \to 0$.*

**Questions.** We conclude with some natural questions suggested by the analogy with hyperbolic surfaces.

1. Let $C$ be a simple cycle. Is the function $L(C, f)$ free of critical points in $\mathcal{B}_d$?

2. Let $(C_i)$ be a binding collection of cycles. Does $\sum L(C_i, f)$ achieve its minimum at a unique point $f \in \mathcal{B}_d$?

3. Let $\mathcal{QB}_d$ denote the space rational maps of the form

$$f(z) = z \prod_1^{d-1} \left( \frac{z - a_i}{1 - b_i z} \right)$$

   such that $\prod |a_i| < 1$, $\prod |b_i| < 1$, and $J(f)$ is a Jordan curve. Each $f \in \mathcal{QB}_d$ can be regarded as a *marked quasiblaschke product*, obtained by gluing together a pair of maps $f_1, f_2 \in \mathcal{B}_d$ using their markings on $S^1$.

   Does there exist an $\epsilon_d > 0$ such for all $f \in \mathcal{QB}_d$, all cycles of length shorter than $\epsilon_d$ are simple?

4. Suppose the cycles $(C_1, C_2)$ are binding. Does the set of $f \in \mathcal{QB}_d$ with $L(C_1, f_1) + L(C_2, f_2) \leq M$ have compact closure in the moduli space of all rational maps of degree $d$?

The analogous questions for hyperbolic surfaces and quasifuchsian groups are known to have positive answers [Ker, §3], [Ot], [Th, Thm 4.4].

**Notes and references.** This paper is a sequel to [Mc4] and [Mc5], which construct a Weil-Petersson metric on $\mathcal{B}_d$ and an embedding of $\mathcal{B}_d$ into the space of invariant measures for $p_d(z) = z^d$.

Simple cycles in degree two play a central role in the combinatorics of the Mandelbrot set [DH], [Ke], and are studied for higher degree in [Gol] and [GM]. Extremal length arguments similar to those we use in §5 are well-known both in the theory of Kleinian groups [Bers, Thm. 3], [Th, Proposition 1.3], [Mc1, §6.3], [Pet1], [Mil2] and rational maps [Pom], [Lev], [Hub], [Pet2]. The quotient Riemann surface of a general rational map is discussed in [McS]; other aspects of the dictionary between rational maps and Kleinian groups are presented in [Mc2]. See [PL] for a related discussion of spinning degenerations of the quotient torus.

## 2 Simple cycles

In this section we discuss the combinatorics of periodic cycles for the map $p_d(t) = d \cdot t \bmod 1$, and prove the closure of the simple cycles has Hausdorff dimension zero.

**Degree and rotation number.** Let $S^1 = \mathbb{R}/\mathbb{Z}$. Given $a \neq b \in S^1$, let $[a, b] \subset S^1$ denote the unique subinterval that is positively oriented from $a$ to $b$. We write $a < c < b$ if $c \in [a, b]$. The length of an interval is denoted $|I|$.

Let $f : S^1 \to S^1$ be a topological covering map of degree $d > 0$, and suppose $f(X) = X$. The *degree* of $f|X$, denoted $\deg(f|X)$, is the least $e > 0$ such that $f|X$ extends to a topological covering $g : S^1 \to S^1$ of degree $e$.

Note that $\deg(f|X) = 1$ iff $f$ preserves the cyclic ordering of $X$, in which case $f|X$ also has a well-defined *rotation number* $\rho(f|X) \in S^1$. If $X$ is finite then $\rho(f|X) = p/q$ is rational and the orbits of $f|X$ have size $q$.

**Example:** Suppose $X = \{x_0, x_1, \ldots, x_n = x_0\}$ in increasing cyclic order, and $f|X$ is a permutation; then we have

$$\deg(f|X) = \sum_0^{n-1} |[f(x_i), f(x_{i+1})]|.$$

Indeed, an extension of $f|X$ of minimal degree is obtain by mapping $[x_i, x_{i+1}]$ homeomorphically to $[f(x_i), f(x_{i+1})]$. The degree is thus a variant of the number of *descents* of a permutation (see e.g. [St, §1.3]).

**The model map and its modular group.** Now fix $d > 1$, and let $p_d(t) = d \cdot t \bmod 1$. Any expanding map $f : S^1 \to S^1$ of degree $d$ is topologically conjugate to $p_d$ [Sh].

The *modular group* $\mathrm{Mod}_d \subset \mathrm{Aut}(S^1)$ is the cyclic group of rotations generated by $t \mapsto 1/(d-1) + t \bmod 1$; it coincides with the group of (degree one) topological automorphisms of $p_d$. Note that $\mathrm{Mod}_d$ acts transitively on the fixed points of $p_d$.

**Simple cycles.** A finite set $C \subset S^1$ is a *cycle* of degree $d$ if $p_d|C$ is a transitive permutation. As in §1, we say a cycle is *simple* if $\deg(p_d|C) = 1$. Simple cycles $(C_1, \ldots, C_m)$ are *compatible* if $\deg(p_d| \bigcup C_i) = 1$.

It is elementary to see:

**Proposition 2.1** *The simple cycles $(C_1, \ldots, C_m)$ are compatible iff they are pairwise compatible.*

We let $\mathcal{C}_d$ denote the set of all cycles of degree $d$, and $\mathcal{C}_d(p/q) \subset \mathcal{C}_d$ the simple cycles with rotation number $p/q$.

**Portraits of fixed points.** The *fixed-point portrait* [Gol] of a simple cycle $C \in \mathcal{C}_d(p/q)$ is the monotone increasing function

$$\sigma : \{1, \ldots, d-2\} \to \{0, 1, \ldots, q\}$$

given by

$$\sigma(j) = |C \cap [0, j/(d-1))|.$$

This invariant specifies how $C$ is interleaved between the fixed points of $p_d$, which are all of the form $j/(d-1) \bmod 1$.

**Basic properties.** The following results are immediate from [Gol] (see especially Lemma 2 and Theorem 7).

1. A simple cycle $C \in \mathcal{C}_d(p/q)$ is uniquely determined by its fixed-point portrait $\sigma(j)$, and all possible monotone increasing functions $\sigma(j)$ arise.
2. The number of simple cycles of degree $d$ and rotation number $p/q$ is $\binom{d+q-2}{q}$.
3. The number of cycles of period $q$ grows like $d^q$, while the number of simple cycles is $O(q^{d-1})$; so most cycles are not simple.
4. Cycles $C_1, C_2 \in \mathcal{C}_d(p/q)$ are compatible iff their fixed-point portraits satisfy
$$\sigma_1(j) \leq \sigma_2(j) \leq \sigma_1(j) + 1$$
for $0 \leq j \leq d-2$, or the same with $\sigma_1$ and $\sigma_2$ reversed.
5. Every maximal collection of compatible cycles has cardinality $d-1$.

**From portraits to cycles.** A simple cycle $C \in \mathcal{C}_d$ can be reconstructed explicitly from its rotation number $p/q$ and its fixed-point portrait $\sigma$ as follows. Let $\tau$ be the 'transpose' of $\sigma$, namely the monotone function $\tau : \{0, 1, \ldots, q-1\} \to \{0, 1, \ldots, d-1\}$ given by

$$\tau(i) = |\{j \,:\, \sigma(j) \leq i\}|, \tag{2.1}$$

and let

$$\tau'(i) = \tau(i) + \begin{cases} 0 & \text{if } 0 \leq i < q-p, \text{ and} \\ 1 & \text{otherwise,} \end{cases}$$

where $i$ is taken mod $q$. Then the periodic point given by $t = 0.\tau'(0)\tau'(p)\tau'(2p)\ldots$ in base $d$ generates $C$; indeed, $t$ is the 'first point' in the cycle $C$.

**Examples.** To simplify notation, let $(p_1/q, \ldots, p_m/q) = (p_1, \ldots, p_m)/q$, and let $\sigma = n_1 \ldots n_{d-1}$ denote the function with values $\sigma(j) = n_j$.

*Degree* $d = 2$. In the quadratic case, $\sigma$ is trivial and hence there is a unique simple cycle $C(p/q)$ for each possible rotation number; e.g.

$$\begin{aligned} C(1/2) &= (1,2)/3, \\ C(1/3) &= (1,2,4)/7, \\ C(2/5) &= (5,10,20,9,18)/31. \end{aligned}$$

The only cycle of period $\leq 4$ which is not simple is $C = (1,2,4,3)/5$. For period 5 there are two such, namely $C$ and $-C$ where $C = (3,6,12,24,17)/31$. Any two distinct quadratic simple cycles are incompatible.

*Degree* $d = 3$. In the cubic case $p_d$ has two fixed points, 0 and 1/2, and three cycles of period two, given by

$$\begin{aligned} C(1/2,0) &= (5,7)/8, \\ C(1/2,1) &= (1,3)/4 \quad \text{and} \\ C(1/2,2) &= (1,3)/8. \end{aligned}$$

The first and last are incompatible, while the other pairs are compatible. In general there are $q+1$ cubic simple cycles with rotation number $p/q$, whose fixed-point portraits are given by $\sigma(1) = 0, 1, \ldots, q$. Only the pairs with adjacent values of $\sigma(1)$ are compatible.

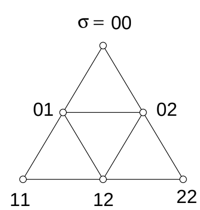


Figure 3. Compatibility of degree 4 cycles of the form $C(1/2, \sigma)$.

*Degree* $d = 4$. In the quartic case there are six cycles in $\mathcal{C}_4(1/2)$, generated by $t = p/15$ with $p = 1, 2, 3, 6, 7$ and 11. The compatibility relation between these cycles is shown in Figure 3. The 4 visible triangles give the 4 distinct triples of compatible simple cycles with rotation number 1/2. Note that the modular group $\mathrm{Mod}_4 \cong \mathbb{Z}/3$ acts by rotations on this diagram.

In general $\mathcal{C}_d(p/q)$ can be identified with the vertices of the $q$-fold barycentric subdivision of a $(d-2)$-simplex, with the top-dimensional cells corresponding to maximal collections of compatible cycles.

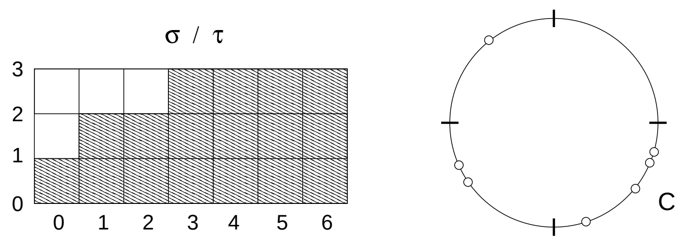


Figure 4. The degree 5 simple cycle with rotation number $3/7$ and $\sigma = 013$.

*Sample computation in degree $d = 5$.* To compute $C(3/7, 013)$, we first use equation (2.1) to compute the 'transpose' $\tau = 1223333$ of $\sigma = 013$. Note that the graphs of $\sigma$ and $\tau$, shown in white and black in Figure 4, fit together to form a rectangle. Evaluating $\tau' = 1223444$ along the sequence $ip \bmod q$, $i = 0, 1, 2, \ldots$ we obtain the base 5 expansion $t = 0.\overline{1342424}_5 = 6966/19531$ for a generator of $C$.

The cycle $C$, along with the 4 fixed points of $p_5$, is drawn at the right in Figure 4. Note that $\sigma = 013$ gives the running total of the number of points of $C$ in the first three quadrants.

**Comparison with simple geodesics.** The simple cycles for $p_d|S^1$ behave in many ways like simple closed geodesics on a compact hyperbolic surface $X = \Delta/\Gamma$ of genus $g$, with compatible cycles corresponding to disjoint geodesics. For example, every maximal collection of disjoint simple closed curves on $X$ has $3g - 3$ elements, just as every maximal collection of compatible cycles for $p_d$ has $d - 1$ elements.

It is also known that the endpoints of lifts of simple geodesics lie in a closed set $E \subset S^1$ of Hausdorff dimension zero [BS]. The analogous statement for simple cycles is:

**Theorem 2.2** *The closure $E$ of the union of all simple cycles $C \subset S^1$ of degree $d$ has Hausdorff dimension zero.*

**Proof.** Let us say a finite set $P \subset S^1$ is a *precycle* if it is the forward orbit of preperiodic point $x \in S^1$ under $p_d$. We say $P$ is *simple*, with rotation number $p/q$, if $p_d|P$ extends to a continuous, monotone increasing map $f : S^1 \to S^1$ with rotation number $p/q$. Then $q \leq n$ and the periodic part $C$ of $P$ is a simple cycle.

Let $\mathcal{P}_d(n, p/q)$ denote the set of all simple precycles of length $n$ and rotation number $p/q$. The argument that shows $|\mathcal{C}_d(p/q)| = O(q^{d-2})$ can be adapted to show that $|\mathcal{P}_d(n, p/q)| = O(n^{d-2})$ as well.

Now fix $N > 0$. We claim that every $x \in E$ lies within distance $O(d^{-N})$ of a simple precycle $P$ with $|P| \leq N$. To find this precycle, simply increase $x$ continuously until two of the points among $x, f(x), \ldots, f^N(x)$ coincide. This requires moving $x$ only slightly, since $|(f^N)'(x)| = d^N$.

Thus $E$ is contained in a neighborhood of diameter $O(d^{-N})$ of the union $E_N$ of all simple precycles with $|P| \leq N$. Since $|E_N| = O(N^{d+2})$ grows only like a polynomial in $N$, this implies $\dim(E) = 0$. ∎

**Proof of Theorem 1.4.** The Hölder continuous conjugacy $\phi_f$ between $f$ and $p_d$ preserves sets of Hausdorff dimension zero. ∎

**Remark: Invariant measures.** The basic properties of simple cycles can also be developed using the correspondence between invariant measures and covering relations established in [Mc5]. For example, any union $D = \bigcup C_i$ of compatible cycles in $\mathcal{C}_d(p/q)$ arises as the support of an invariant measure $\nu$ for $p_d|S^1$. Invariant measures, in turn, correspond bijectively to covering relations $(F, S)$ of degree $d$. In the case at hand, $F(t) = t + p/q \bmod 1$ and $S$ is a divisor on $S^1$ of degree $d-1$. By perturbing $S$ so its points have multiplicity one, we obtain a nearby invariant measure $\nu'$ whose support $D' \supset D$ is a maximal union of exactly $(d-1)$ compatible cycles (property (5) above).

The compactification of the space of Blaschke products by covering relations $(F, S)$ is discussed in the following section.

**Question.** Is there a useful notion of intersection number for a pair of cycles?

## 3 Blaschke products

This section presents basic facts about marked Blaschke products, their derivatives and their images in the moduli space of all rational maps. See [Mc5] for related background material.

**Blaschke products.** Identify $S^1 = \mathbb{R}/\mathbb{Z}$ with the unit circle in the complex plane, using the coordinate $z = \exp(2\pi i t)$. Let $\Delta = \{z : |z| < 1\}$ be the unit disk, and $\Delta^{(n)}$ its $n$-fold symmetric product.

Given $d > 1$, let $\mathcal{B}_d \cong \Delta^{(d-1)}$ denote the space of Blaschke products $f : \Delta \to \Delta$ of the form

$$f(z) = z \prod_1^{d-1} \left( \frac{z - a_i}{1 - \overline{a}_i z} \right)$$

with $a_i \in \Delta$. Note that $f$ extends to a rational map on the whole Riemann sphere, and $f|S^1$ is a covering map of degree $d$.

A proper holomorphic map $g : \Delta \to \Delta$ of degree $d > 1$ is conjugate to some $f \in \mathcal{B}_d$ iff $g$ has a fixed point.

**Derivatives and measure.** By logarithmic differentiation, any $f \in \mathcal{B}_d$ satisfies

$$|f'(z)| = 1 + \sum_1^{d-1} \frac{1 - |a_i|^2}{|z - a_i|^2} \tag{3.1}$$

for $z \in S^1$. In particular, $f|S^1$ is expanding.

More importantly, $f|S^1$ preserves normalized Lebesgue measure $\lambda$ on the circle; equivalently, $f_*(dz/z) = dz/z$, as can be verified by residue considerations. This means

$$\sum_{f(w)=z} |f'(w)|^{-1} = 1 \tag{3.2}$$

for any $z \in S^1$.

**Markings.** All $f \in \mathcal{B}_d$ are topologically conjugate to the model mapping $p_d(z) = z^d$. A *marking* for $f$ the choice of one such conjugacy, i.e. the choice of a degree one homeomorphisms $\phi : S^1 \to S^1$ such that

$$f(z) = \phi^{-1} \circ p_d \circ \phi(z).$$

There is a unique marking $\phi_f$ which varies continuously in $f$ and satisfies $\phi_f(z) = z$ when $f = p_d$. Thus $\mathcal{B}_d$ can be regarded as the space of *marked Blaschke products*.

The modular group $\mathrm{Mod}_d \cong \mathbb{Z}/(d-1)$ acts on $\mathcal{B}_d$ by $(a_i) \mapsto (\zeta a_i)$ where $\zeta^{d-1} = 1$. Its orbits correspond to different markings of the same map. Thus $f_1, f_2 \in \mathcal{B}_d$ are conformally conjugate on $\Delta$ iff they are in the same orbit of the modular group.

**Lengths.** The canonical marking allows one to label the cycles of $f$ by the cycles of $p_d$. We define the *length on* $f$ of a cycle $C \in \mathcal{C}_d$ of period $q$ by

$$L(C, f) = \log |(f^q)'(z)|$$

for any $z \in S^1$ with $\phi_f(z) \in C$.

**Limits of lower degree.** The space of Blaschke products has a natural compactification $\overline{\mathcal{B}}_d \cong \overline{\Delta}^{(d-1)}$, whose boundary points $(a_i)$ can be interpreted as pairs $(F, S)$ consisting of a Blaschke product

$$F(z) = z \prod_{|a_i|<1} \left( \frac{z - a_i}{1 - \overline{a}_i z} \right) \cdot \prod_{|a_i|=1} (-a_i)$$

and a divisor of *sources*

$$S = \sum_{|a_i|=1} 1 \cdot a_i \in \mathrm{Div}(S^1),$$

satisfying $\deg F + \deg S = d$. It is easy to see:

**Proposition 3.1** *A sequence $f_n \in \mathcal{B}_d$ converges to $(F, S) \in \partial\mathcal{B}_d$ iff*

> *(i) $f_n(z) \to F(z)$ uniformly on compact subsets of $\widehat{\mathbb{C}} - \operatorname{supp} S$, and*
> *(ii) the zeros $Z(f_n)$ converge to $Z(F) + S$ as divisors on $\widehat{\mathbb{C}}$.*

More generally, the space $\mathrm{Rat}_d$ of degree $d$ rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has a compactification $\overline{\mathrm{Rat}_d} \cong \mathbb{P}^{2d+1}$, whose boundary points $(F, S)$ are pairs consisting of a rational map $F$ and an effective divisor $S \in \mathrm{Div}(\widehat{\mathbb{C}})$ with $\deg(F) + \deg(S) = d$. We have $f_n \to (F, S)$ in $\overline{\mathrm{Rat}_d}$ iff their graphs satisfy

$$\mathrm{gr}(f_n) \to \mathrm{gr}(F) + S \times \widehat{\mathbb{C}}$$

as divisors of degree $(1, d)$ on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ (cf. [D, §1]).

**Radial bounds on $f'(z)$.** The following elementary observation is useful for studying limits as above.

**Proposition 3.2** *For any proper holomorphic map $f : \Delta \to \Delta$ and $\zeta \in S^1$, we have*

$$\sup_{r \in [0,1]} |f'(r\zeta)| \le 4|f'(\zeta)|.$$

Note that we do not require that $f(0) = 0$. This bound is sharp, as can be seen by considering $f(z) = (z + a)/(1 + az)$ as $a \to 1-$.

**Proof.** We can write

$$f(z) = e^{i\theta} \prod_1^d M_i(z), \tag{3.3}$$

where $M_i(z) = (z - a_i)/(1 - \overline{a}_i z)$ and $a_i \in \Delta$. Composing with a rotation, we can also assume that $\zeta = 1$. For $r \in [0, 1]$ we have

$$\left| \frac{M_i'(r)}{M_i'(1)} \right| = \frac{|1 - a_i|^2}{|1 - ra_i|^2},$$

and therefore

$$|M_i'(r)| \le 4|M_i'(1)|,$$

since the distance from 1 to $a_i$ is never more than twice the distance from 1 to $ra_i$, Differentiating the product (3.3) and using the fact that $|\prod_{j \neq i} M_j(r)| \le 1$, we obtain:

$$|f'(r)| \le \sum |M_i'(r)| \le 4 \sum |M_i'(1)| = 4|f'(1)|.$$

The last equality, like equation (3.1), is verified by logarithmic differentiation. ∎

**Corollary 3.3** *If $f_n \to (F, S) \in \overline{\mathcal{B}}_d$, $z_n \in S^1$, $z_n \to z$ and $|f_n'(z_n)| = O(1)$, then* $\lim f_n(z_n) = F(z)$.

**Proof.** Suppose $\sup |f_n'(z_n)| = M$; then for any $r < 1$ we have

$$\begin{aligned} \limsup |f_n(z_n) - F(z)| &\le \limsup |f_n(rz_n) - F(z)| + 4M(1 - r) \\ &= |F(rz) - F(z)| + 4M(1 - r); \end{aligned}$$

now let $r \to 1$. ∎

**Irrational rotations.** As a sample application, we prove the following result stated in the Introduction:

**Corollary 3.4** *If $f_n \in \mathcal{B}_d$ satisfies $f_n'(0) \to \exp(2\pi i\theta)$ where $\theta$ is irrational, then $L(C, f_n) \to \infty$ for every cycle $C$.*

**Proof.** Suppose to the contrary that $L(C, f_n)$ is bounded for some cycle $C$. Let $C_n \subset S^1$ be the corresponding periodic cycle for $f_n$. Pass to a subsequence such that $f_n \to (F, S) \in \partial\mathcal{B}_d$ and $C_n \to D \subset S^1$ in the Hausdorff topology. Then $F(z) = \exp(2\pi i\theta)z$ and by Corollary 3.3 we have $F(D) = D$, contradicting the irrationality of $\theta$. ∎

**Variants.** Here are two useful variants of the results above:

**Proposition 3.5** *For any proper holomorphic map $f : \mathbb{H} \to \mathbb{H}$ and $x \in \mathbb{R}$, we have*

$$\sup_y |f'(x+iy)| \le f'(x).$$

**Proposition 3.6** *Assume $f_n \in \operatorname{Rat}_d$ converges to $(F,S) \in \overline{\operatorname{Rat}}_d$, $z_n \to z$, and $\|Df_n(z_n)\| = O(1)$ in the spherical metric on $\widehat{\mathbb{C}}$. Then we have*

$$f_n(z_n) \to F(z)$$

*provided $z_n$ belongs to a circle $T_n$ with $f_n^{-1}(T_n) = T_n$, and $\inf_n \operatorname{diam}(T_n) > 0$.*

**Proofs.** The first result follows directly from the representation $f(z) = a_0 z + b_0 + \sum_1^{d-1} a_i/(b_i - z)$ with $a_i > 0$ and $b_i \in \mathbb{R}$, and the second follows by the same argument as Corollary 3.3. ∎

The maps $f_n(z) = 1/(1+nz^2)$ satisfy $f_n'(0) = 0$ and $\lim f_n(0) = 1 \neq F(0) = 0$; thus some extra hypothesis is needed to interchange limits as in Proposition 3.6.

**Moduli space of rational maps.** Let $\operatorname{MRat}_d = \operatorname{Rat}_d / \operatorname{Aut}(\widehat{\mathbb{C}})$ denote the moduli space of holomorphic conjugacy classes of rational maps of degree $d > 1$. A pair of Blaschke products are conjugate iff they are related by the modular group or by $z \mapsto 1/z$; thus we have an inclusion

$$\mathcal{B}_d / (\operatorname{Mod}_d \ltimes \mathbb{Z}/2) \hookrightarrow \operatorname{MRat}_d .$$

The next result shows this inclusion is almost proper.

**Theorem 3.7** *If $f_n \to (F,S) \in \partial\mathcal{B}_d$ but $[f_n]$ remains bounded in $\operatorname{MRat}_d$, then $F(z) = z$ and $\operatorname{supp} S$ is a single point. In particular, we have $f_n'(0) \to 1$.*

**Proof.** Pass to a subsequence such $[f_n] \to [g] \in \operatorname{MRat}_d$ and $f_n \to (F,S) \in \partial\mathcal{B}_d$. Then there are conjugates $h_n = A_n f_n A_n^{-1} \to g$. Since $f_n$ diverges in $\mathcal{B}_d$, $A_n \to \infty$ in $\operatorname{Aut}(\widehat{\mathbb{C}})$. On the other hand, the measures of maximal entropy satisfy $\mu(h_n) \to \mu(g)$ and $\mu(f_n) \to \mu(F,S)$, by [D, Thm. 0.1] (see also [Mc5]). Since $\mu(g)$ is nonatomic, this implies $\mu(F,S) = \lim A_n^*(\mu(h_n))$ is supported at a single point. But $\operatorname{supp} \mu(F,S)$ is $F$-invariant and includes $\operatorname{supp} S$; thus $F(z) = z$ and $\operatorname{supp} S = \{s\}$ is itself a single point. ∎

**Example.** The sequence $f_n(z) = z(z + a_n)/(1 + a_n z)$, with $a_n = 1 - 1/n$, is divergent in $\mathcal{B}_2$ but convergent in $\operatorname{MRat}_2$. To see this, normalize so the origin is a critical point instead of a fixed point; then $f_n(z)$ is conjugate to $h_n(z) = (z^2 + b_n)/(1 + b_n z^2)$, and $b_n = a_n/(2 + a_n) \to 1/3$ as $a_n \to 1$.

## 4 The thin part of $f(z)$

Let us define the *thin part* of $f \in \mathcal{B}_d$ by

$$S^1_{\text{thin}}(f) = \{z \in S^1 \ : \ |f'(z)| < 2\}.$$

In this section we will show:

**Theorem 4.1** *For any $f \in \mathcal{B}_d$, the map $f|S^1_{\text{thin}}(f)$ extends to a degree one homeomorphism of the circle.*

**Corollary 4.2** *All cycles of $f$ with $L(C,f) < \log 2$ are simple and compatible.*

**Visual angles.** The derivative of

$$f(z) = z \prod_1^{d-1} \left( \frac{z - a_i}{1 - \overline{a}_i z} \right)$$

can be conveniently analyzed using the *hyperbolic visual angle*, defined for $a, z \in \overline{\Delta}$ by

$$\alpha(z,a) = 2\arg(z-a) - \arg(z).$$

This is the angle at $a$ of the hyperbolic geodesic $\overline{az}$. For $z \in S^1$ we have $\arg(1 - \overline{a}z) = \arg(z) - \arg(z-a)$, and thus

$$\arg(f(z)) = \arg(z) + \sum_1^{d-1} \alpha(z, a_i). \tag{4.1}$$

(Note this simplifies to $\arg(f(z)) = 2\arg(z - a_1)$ when $d = 2$.) Letting $\theta = \arg(z)$ and $\dot{\alpha} = d\alpha/d\theta$, we then obtain:

$$|f'(z)| = 1 + \sum_1^{d-1} \dot{\alpha}(z, a_i) \tag{4.2}$$

for $z \in S^1$.

**The visual density.** The *visual density* $\dot{\alpha}(z,a)$ is essentially the Poisson kernel; for $a = r \geq 0$ it is given by

$$\dot{\alpha}(z,r) = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}, \tag{4.3}$$

where $\theta = \arg z$. Geometrically, $(\dot{\alpha}(z,a)/2\pi)\,d\theta$ is the hitting measure on the circle for a random hyperbolic geodesic starting at $a$.

For fixed $z \in S^1$, the level sets of $\dot{\alpha}(z,a)$ are horocycles resting on $z$. Thus

$$J(a) = \{z \in S^1 \, : \, \dot{\alpha}(z,a) < 1\}$$

is the large arc cut off by the chord perpendicular to $\overline{0a}$. This follows from the fact that the horocycle resting on one of the endpoints of $J(a)$ and passing through 0 also passes through $a$ (see Figure 5).

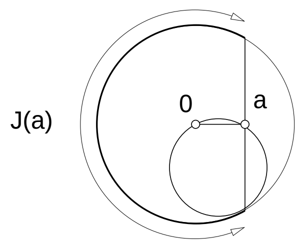


Figure 5. The arc $J(a)$ where $\dot{\alpha}(z,a) < 1$.

**Proposition 4.3** *The visual density $\dot{\alpha}(z,a)|J(a)$ is strictly convex, and decreases as $a$ moves radially towards the circle. In other words, we have*

$$\dddot{\alpha}(z,a) > 0 \quad \textit{and} \quad \frac{d}{ds}\dot{\alpha}(z,sa)\bigg|_{s=1} < 0$$

*for all $z \in J(a)$.*

**Proof.** To verify convexity, consider the case where $a = r \in [0,1)$. By (4.3), in this case we have $\dot{\alpha} = (1-r^2)/u$ where $u = 1 + r^2 - 2r\cos\theta$. We may assume $\theta \in (0,\pi)$. Cross-multiplying and differentiating, we obtain

$$\begin{aligned}
&\dot{\alpha}u = 1 - r^2, \\
&\ddot{\alpha}u + \dot{\alpha}(2r\sin\theta) = 0, \quad \text{and} \\
&\dddot{\alpha}u + \ddot{\alpha}(4r\sin\theta) + \dot{\alpha}(2r\cos\theta) = 0.
\end{aligned}$$

Since $r, u$ and $\sin\theta$ are all positive, we have $\dot{\alpha} > 0$ and $\ddot{\alpha} < 0$. Comparing the last two equations, we find the sign of $\dddot{\alpha}$ is the same as the sign of the determinant

$$D = \det\begin{pmatrix} 2r\sin\theta & u \\ 2r\cos\theta & 4r\sin\theta \end{pmatrix} = 8r^2\sin^2\theta - 2ru\cos\theta.$$

We claim $D > 0$ when $z \in J(r)$, i.e. when $u = |z - r|^2 > 1 - r^2$. The claim is evident if $\cos\theta$ is negative, so assume $\theta \in (0, \pi/2)$; then

$$u = |z - r|^2 \leq |z - 1|^2 \leq 2(\operatorname{Im} z)^2 = 2\sin^2\theta.$$

We also have $\cos\theta = \operatorname{Re}(z) < r$ for $z \in J(r)$, and thus:

$$D \geq 4r^2 u - 2r^2 u > 0.$$

The proof of the density decreasing property is straightforward. ∎

**Properties of the thin part of *f*.** We can now show that $f|S^1_{\text{thin}}(f)$ acts like a rotation. We first observe:

**Proposition 4.4** *For any $f \in \mathcal{B}_d$,*

*(i) The map $f|S^1_{\text{thin}}(f)$ is injective,*

*(ii) We have $S^1_{\text{thin}}(f) \subset \bigcap J(a_i)$ ,*

*(iii) $S^1_{\text{thin}}(f)$ consists of at most $(d-1)$ disjoint open intervals, and*

*(iv) $S^1_{\text{thin}}(f)$ increases as the zeros $a_i$ of $f$ move radially towards the circle.*

**Proof.** If $f(x_1) = f(x_2)$ for two distinct points in $S^1_{\text{thin}}(f)$, then $|f'(x_1)| + |f'(x_2)| > 1/2 + 1/2 = 1$, which violates the measure-preserving property (3.2) of $f$; thus $f|S^1_{\text{thin}}(f)$ is injective. Equation (4.2) implies (ii). Since $\bigcup(S^1 - J(a_i))$ has at most $(d-1)$ components, so does $I = \bigcap J(a_i)$. By Proposition 4.3, $|f'(z)|$ is locally convex on $I$; thus the intersection of $S^1_{\text{thin}}(f)$ with any component of $I$ is connected, and (iii) follows. The density decreasing property stated in Proposition 4.3 implies (iv). ∎

**Proof of Theorem 4.1.** By moving the points $(a_i)$ radially to the circle, we obtain a smooth 1-parameter family of maps $f_t \in \overline{\mathcal{B}}_d$, $t \in [0,1]$, with $f_0 = f$ and $f_1 = (F, S)$. Since $\deg(S) = d - 1$, we have $\deg(F) = 1$. Proposition 4.4 implies that $f_t|T_t = S^1_{\text{thin}}(f_t)$ is injective, $T_s \subset T_t$ when $s < t$, and $\operatorname{supp} S \cap T_t = \emptyset$. Thus for any three distinct points $x_i \in S^1_{\text{thin}}(f)$, the triple $(f_t(x_1), f_t(x_2), f_t(x_3))$ moves by isotopy as $t$ increases from 0 to 1, and converges to $F(x_1), F(x_2), F(x_3))$ as $t \to 1$. Since $F$ is a rotation, it preserves the cyclic ordering of the points $(x_i)$, so the same is true of $f$. Consequently $f$ extends from $S^1_{\text{thin}}(f)$ to an orientation-preserving homeomorphism of the circle. ∎

## 5 Bounds on repelling cycles

In this section we show that every $f \in \mathcal{B}_d$ has a simple cycle with $L(C, f) = O(d)$, and obtain related results for general rational maps.

**Moduli and tori.** We begin by summarizing some well-known facts about extremal length on tori.

Any point $\tau \in \mathbb{H}$ determines a complex torus

$$X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$$

with a flat metric inherited from the plane, and a distinguished basis $\langle 1, \tau \rangle$ for its fundamental group. Factoring the covering map $\mathbb{C} \to X_\tau$ through the map $\xi : \mathbb{C} \to \mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$ given by $\xi(z) = \exp(2\pi i z)$, we have

$$X_\tau = \mathbb{C}^*/\alpha^{\mathbb{Z}}$$

where $\alpha = \xi(\tau)$ satisfies $0 < |\alpha| < 1$. The same construction can be made when $-\tau \in \mathbb{H}$; then $|\alpha| > 1$.

Given a slope $p/q \in \mathbb{Q} \cup \{\infty\}$, let $\gamma_{p/q} \subset X_\tau$ denote the simple closed geodesic obtained as the projection of the line $\mathbb{R} \cdot (\tau - p/q)$ from $\mathbb{C}$ to $X_\tau$. Its preimage $\widetilde{\gamma}_{p/q}$ in the intermediate cover $\mathbb{C}^*$ consists of $q$ arcs joining 0 to $\infty$, cyclically permuted with rotation number $p/q$ by $z \mapsto \alpha z$.

Any annulus $A$ is conformally equivalent to a right cylinder, which is unique up to scale. The ratio $\operatorname{mod}(A) = h/c$ between the height and circumference of this cylinder is the *modulus* of $A$.

The maximum modulus of an annulus $A \subset X_\tau$ homotopic to $\gamma_{p/q}$ is given by

$$\operatorname{mod}(p/q, X_\tau) = \frac{\operatorname{area}(X_\tau)}{L(\gamma_{p/q}, X_\tau)^2} = \frac{|\operatorname{Im}\tau|}{|q\tau - p|^2} \tag{5.1}$$

(assuming $\gcd(p, q) = 1$). This maximum is realized by taking $A = X_\tau \setminus \gamma_{p/q}$. The set of $\tau \in \mathbb{H}$ with $\operatorname{mod}(p/q, X_\tau) \geq m$ is a horoball of diameter $1/(mq^2)$ resting on the real axis at $p/q$. For $p/q = 1/0$ we have

$$\operatorname{mod}(\infty, X_\tau) = |\operatorname{Im}\tau|.$$

The *intersection inequality*

$$\operatorname{mod}(p/q, X_\tau)\operatorname{mod}(r/s, X_\tau) \leq \left(\det \begin{pmatrix} p & q \\ r & s \end{pmatrix}\right)^{-2} \tag{5.2}$$

is easily verified by considering the determinant of the lattice $\mathbb{Z}(q\tau - p) \oplus \mathbb{Z}(s\tau - r)$. This inequality implies:

*There is at most one slope with* $\operatorname{mod}(p/q, X_\tau) > 1$.

On the other hand we have:

**Proposition 5.1** *For any* $\tau \in \mathbb{H}$*, there exists a slope* $p/q \in \mathbb{Q} \cup \{\infty\}$ *such that*

$$\operatorname{mod}(p/q, X_\tau) \geq \sqrt{3}/2.$$

**Proof.** Since the statement is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, it suffices to verify it when $\tau$ lies in the fundamental domain $|\tau| \geq 1$, $|\operatorname{Re}\tau| \leq 1/2$; and in this case, we have $\operatorname{mod}(\infty, X_\tau) = \operatorname{Im}\tau \geq \sqrt{3}/2$. ■

**Rational maps.** Now let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d > 1$. If $z \in \widehat{\mathbb{C}}$ is a point of period $q$, its *multiplier* is given by $\beta = (f^q)'(z)$. The *grand orbit* of $z$ is the set $\bigcup_{i,j>0} f^{-i} \circ f^j(z)$.

Suppose $f$ has a fixed point at $z = 0$ and a periodic point $w \neq 0$ with period $q$. We say $w$ has *rotation number* $p/q$ relative to $z = 0$ if there are arcs $(\delta_i)_0^{q-1} \subset \widehat{\mathbb{C}}$ joining $z = 0$ to $f^i(w)$, meeting only at $z = 0$, which are cyclically permuted by $f$ with rotation number $p/q$.

**Theorem 5.2** *Let* $f$ *be a rational map with an attracting fixed-point at* $z = 0$*, with multiplier*

$$\alpha = f'(0) = \exp(2\pi i \tau) \neq 0.$$

*Let* $e$ *be the number of grand orbits of critical points in the immediate basin* $\Omega$ *of* $z = 0$*. Then for each* $p/q \in \mathbb{Q}$*, there exists a repelling or parabolic periodic point* $w \in \partial\Omega$ *such that:*

1. *The rotation number of* $w$ *relative to* $z = 0$ *is* $p/q$*; and*
2. *Its multiplier has the form* $\beta = (f^q)'(w) = \exp(-2\pi i \sigma)$*, where* $\sigma = 0$ *or*
$$\frac{\operatorname{Im}\sigma}{|\sigma|^2} \geq \frac{\operatorname{mod}(p/q, X_\tau)}{e}. \tag{5.3}$$

*In particular, we have*

$$|\beta| \leq \left(\exp\left(\frac{2\pi}{\operatorname{mod}(p/q, X_\tau)}\right)\right)^e. \tag{5.4}$$

**Proof.** Let $\Omega^*$ denote the immediate basin of $z = 0$ with the grand orbits of all critical points in $\Omega$ and of $z = 0$ deleted. Then $f : \Omega^* \to \Omega^*$ is a covering map. Moreover, the holomorphic linearizing map

$$\phi(z) = \lim \alpha^{-n} f^n(z)$$

is defined for all $z \in \Omega^*$, and satisfies $\phi(f(z)) = \alpha f(z)$. Consequently $\phi$ descends to an inclusion of the space of grand orbits $Y = \Omega^*/\langle f\rangle$ into the torus $X_\tau = \mathbb{C}^*/\alpha^{\mathbb{Z}}$, making the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\phi} & \mathbb{C}^* \\ \downarrow & & \downarrow \\ \Omega/\langle f\rangle = Y & \hookrightarrow & X_\tau = \mathbb{C}^*/\alpha^{\mathbb{Z}} \end{array}$$

commute. By assumption we have $|Y - X_\tau| = e$.

For a given $p/q \in \mathbb{Q}$, the geodesics parallel to $\gamma_{p/q}$ passing through the punctures of $Y$ cut it into $\leq e$ parallel annuli, one of which satisfies

$$\operatorname{mod}(A) \geq \operatorname{mod}(p/q, X_\tau)/e. \tag{5.5}$$

Let $\delta \subset A$ be the core curve of $A$, and $\delta_0 \subset \Omega^*$ one of its lifts which is incident to $z = 0$. Let $\delta_i = f^i(\delta_0)$. By construction, the arc $\delta_0$ is invariant under $f^q$, and $f^q|\delta_0$ is a bounded translation in the hyperbolic metric on $\Omega^*$. Consequently $\delta_0$ must join $z = 0$ to another fixed point $w$ of $f^q$ in $\partial\Omega$. By the Snail Lemma [Mil1, Lem. 16.2], $w$ is repelling or parabolic.

We have seen that the preimage of $\gamma_{p/q}$ on $\mathbb{C}^*$ consists of $q$ arcs, cyclically permuted with rotation number $p/q$ by $z \mapsto \alpha z$. Since $\phi$ is a homeomorphism near $z = 0$, the arcs $\delta_0, \ldots, \delta_{q-1}$ are also cyclically permuted with rotation number $p/q$ by $f$. In particular $w$ has rotation number $p/q$ relative to $z = 0$.

Now suppose $w$ is repelling, with multiplier $\beta$. Choose an injective branch of $f^{-q}$ defined on a punctured neighborhood $U^*$ of $w$ such that $f^{-q} : U^* \to U^*$ and

$$Z = U^*/\langle f^{-q}\rangle \cong \mathbb{C}^*/\beta^{\mathbb{Z}} = X_\sigma,$$

where $\sigma = \log(\beta/2\pi i)$. There is a unique choice of the logarithm such that the invariant arc $\delta_0 \cap U^*$ descends to a loop isotopic to $\gamma_0$ on $X_\sigma$.

By construction, $A \subset Y$ is covered by a strip $A_0 \subset \Omega^*$ which retracts to $\delta_0$, and hence we have an inclusion

$$A \cong A_0/\langle f^q\rangle \hookrightarrow Z \cong X_\sigma$$

in the same homotopy class as $\gamma_0$. This implies

$$\mathrm{mod}(0, X_\sigma) \geq \mathrm{mod}(A),$$

and the bound (5.3) follows from equations (5.1) and (5.5). ■

**Corollary 5.3** *If $f \in \mathrm{Rat}_d$ has an attracting fixed point with multiplier satisfying*

$$|\alpha| > \exp(-\pi\sqrt{3}) = 0.0043\ldots$$

*then it also has a repelling or parabolic cycle with multiplier satisfying*

$$|\beta| \leq \exp(4\pi/\sqrt{3})^{2d-2} \leq 1416^{2d-2}.$$

**Proof.** The lower bound on $|\alpha|$ implies $\mathrm{Im}(\tau) = \mathrm{mod}(\infty, X_\tau) < \sqrt{3}/2$, where $\tau = (\log\alpha)/2\pi i$. Hence $\mathrm{mod}(p/q, X_\tau) \geq \sqrt{3}/2$ for some $p/q \in \mathbb{Q}$, by Proposition 5.1. Now apply equation (5.4) and note that $e \leq 2d-2$. ■

**Corollary 5.4** *If a map $f \in \mathrm{Rat}_d$ has an attracting fixed point with multiplier $\alpha$, then it also has a repelling or parabolic cycle with multiplier satisfying*

$$|\beta| \leq \Big(\exp(4\pi/\sqrt{3})/|\alpha|\Big)^{2d-2}.$$

**Proof.** Choose $\tau = (\log\alpha)/2\pi i = x+iy$ with $x \in [-1/2, 1/2]$. The previous corollary shows the desired bound holds when $y < \sqrt{3}/2$. For $y \geq \sqrt{3}/2$ we have

$$m = \mathrm{mod}(0, X_\tau)^{-1} \leq \frac{x^2+y^2}{y} \leq \frac{1}{2\sqrt{3}} + y < \frac{2}{\sqrt{3}} + y,$$

which implies $\exp(2\pi/m) \leq \exp(4\pi/\sqrt{3})/|\alpha|$; thus by (5.4) the desired bound holds in this case as well. ■

**The bottom of the spectrum.** Here is a qualitative consequence of the preceding corollary.

Let the *spectrum* $S(f) \subset \mathbb{C}$ be the set of all multipliers $\beta$ that arise from periodic points of $f \in \mathrm{Rat}_d$, and let

$$L(f) = \inf\{\log|\beta| \; : \; \beta \in S(f) \text{ and } |\beta| \geq 1\}.$$

By the fixed-point formula for rational maps [Mil1, Thm. 12.4], the multipliers of $f$ at its fixed points satisfy

$$\sum \frac{1}{\mu_j - 1} = 1, \tag{5.6}$$

provided no $\mu_j = 1$; in particular, $|\mu_j| \le d+1$ for some $j$. Thus if $f$ has no attracting fixed points, it satisfies

$$L(f) \le \log(d+1).$$

Combining this observation with Corollary 5.4, we obtain:

**Corollary 5.5** *Let $f_n \in \mathrm{Rat}_d$ be a sequence of rational maps with $L(f_n) \to \infty$. Then the maps $f_n$ have fixed points with multipliers $\alpha_n \to 0$.*

**Examples.** It is easy to see that $f_n(z) = z^2 + n^2$ satisfies $L(f_n) \to \infty$ as $n \to \infty$, since its Julia set lies close to $\pm n$. Of course $f_n$ has a fixed point at infinity with multiplier $\alpha_n = 0$.

Parabolics must be included in the definition of $L(f)$ to obtain Corollary 5.5. In fact, if we let $L^*(f) = \inf\{\log|\beta| \,:\, \beta \in S(f), |\beta| > 1\}$, then $f_n(z) = z - 1/z + n$ satisfies $L^*(f_n) \to \infty$ even though $f_n$ has no attracting fixed point. (The map $f_n(z)$ behaves like the Hecke group $\langle z \mapsto -1/z, z \mapsto z+n \rangle$; cf. [Mc3, Thm 6.2].)

**Question.** Does Corollary 5.5 remain true if only parabolic and repelling multipliers are included in the definition of $L(f)$?

**Blaschke products.** We now return to the setting of a proper map $f : \Delta \to \Delta$ fixing $z = 0$. In this case formula (5.6) implies:

**Proposition 5.6** *The multipliers $(\lambda_i)_1^{d-1}$ of $f \in \mathcal{B}_d$ at its fixed points on the circle satisfy*

$$\sum_1^{d-1} \frac{1}{\lambda_i - 1} = \frac{1 - |\alpha|^2}{|1-\alpha|^2},$$

*where $\alpha = f'(0)$.*

**Corollary 5.7** *If $|\alpha| < 1/2$, then $f$ has a repelling fixed point with multiplier satisfying $1 < \beta \le 1 + (d-1)/3$.*

**Theorem 5.8** *Every $f \in \mathcal{B}_d$ has a simple cycle with $L(C, f) = O(d)$.*

**Proof.** Combine Corollaries 5.4 and 5.7. ∎

## 6 Short cycles and short geodesics

In this section we use the fixed-point formula for rational maps to obtain the following more detailed connection between the short cycles for $f$ and the short geodesics on its quotient torus.

**Theorem 6.1** *Given $f \in \mathcal{B}_d$ with $f'(0) = \exp(2\pi i\tau)$, choose $p/q \in \mathbb{Q}$ to maximize $\operatorname{mod}(p/q, X_\tau)$. Then there exist compatible simple cycles $C_i$ with rotation number $p/q$, such that:*

1. *Their lengths satisfy*

$$\operatorname{mod}(p/q, X_\tau) \le \pi \sum L(C_i, f)^{-1} \le \operatorname{mod}(p/q, X_\tau) + O(d); \tag{6.1}$$

2. *All other cycles satisfy $L(C, f) > \epsilon_d > 0$; and*
3. *For any $r > 0$, the multipliers of $f^r$ at its repelling fixed points satisfy*

$$\frac{1}{r} \sum{}' \frac{1}{\lambda_j - 1} = O(d), \tag{6.2}$$

*where the prime indicates that fixed points in $\bigcup C_i$ are excluded.*

In qualitative terms, the construction shows:

**Corollary 6.2** *All cycles with $L(C, f) < \epsilon_d$ arise from short geodesics on the quotient torus for $f$.*

**Tiling of $\boldsymbol{\Delta^*}$.** The slope $p/q \bmod 1$ appearing in the Theorem above depends only on $\alpha = f'(0) \in \Delta^*$. Figure 1 of the Introduction shows the regions $T(p/q) \subset \Delta^*$ where a given slope maximizes the value of $\operatorname{mod}(p/q, X_\tau) = \operatorname{mod}(p/q, \mathbb{C}/\alpha^{\mathbb{Z}})$.

This picture is nothing more than the image, under the covering map $\xi : \mathbb{H} \to \Delta^*$ given by $\xi(\tau) = \exp(2\pi i\tau)$, of the tiling of $\mathbb{H}$ by $\mathrm{SL}_2(\mathbb{Z})$ translates of the Dirichlet region

$$F = \{\tau \in \mathbb{H} \,:\, |\tau - n| \ge 1 \;\forall n \in \mathbb{Z}\}$$

for the cusp $\tau = \infty$. The tile $T(\infty) = \xi(F)$ lies in a ball of radius $\exp(-\pi\sqrt{3}) \approx 1/230$ about the origin. In this tile the short curve is $\gamma_\infty \subset X_\tau$, which lifts to a loop around $z = 0$ rather than a path connecting $z = 0$ to a periodic point. Thus the length of $\gamma_\infty$ can go to zero without any cycle getting short.

Each remaining tile $T(p/q)$ contains a horocycle $H$ resting on the root of unity $\exp(2\pi i p/q) \in S^1$. Within a still smaller horocycle $H' \subset H$, $\gamma_{p/q}$ becomes very short, and hence $f$ has a very short cycle with rotation number $p/q$.

**Moduli and multipliers.** We begin the proof of Theorem 6.1 by connecting Diophantine properties of $\alpha \in \Delta^*$ to lengths of geodesics on $\mathbb{C}^*/\alpha^{\mathbb{Z}}$.

**Lemma 6.3** *For any $\alpha = \exp(2\pi i\tau) \in \Delta^*$ and $q > 0$, we have*

$$\sup_p \frac{\operatorname{mod}(p/q, X_\tau)}{\gcd(p,q)^2} = \frac{\pi}{q}\frac{1-|\alpha^q|^2}{|1-\alpha^q|^2} + O(1).$$

**Proof.** First consider the case $q = 1$, and assume $\tau$ is chosen so $|\operatorname{Re}\tau| \leq 1/2$. Then we have $2\pi i\tau \approx 1 - \alpha$ when either side is small, and hence

$$\sup_p \operatorname{mod}(p, X_\tau) = \frac{\operatorname{Im}\tau}{|\tau|^2} = \pi\frac{1-|\alpha|^2}{|1-\alpha|^2} + O(1).$$

The general case follows using the fact that

$$\frac{\operatorname{mod}(p/q, X_\tau)}{\gcd(p,q)^2} = \frac{\operatorname{mod}(p, X_{q\tau})}{q}.$$

∎

**Proof of Theorem 6.1.** Choose $p$ so that $\operatorname{mod}(p/q, X_\tau)$ is maximized. As in Theorem 5.2, by cutting the torus $X_\tau$ open along $e \leq d-1$ geodesics parallel to $\gamma_{p/q}$ we obtain annuli $A_1, \ldots, A_e \subset Y$ with

$$\operatorname{mod}(p/q, X_\tau) = \sum \operatorname{mod}(A_i).$$

Each annulus $A_i$, when lifted to the unit disk, connects $z = 0$ to a simple cycle $C_i$ for $f$ with rotation number $p/q$ and multiplier $\beta_i > 1$.

The lifts of the annuli $A_i$ are disjoint, so the cycles $C_i$ are compatible. Assume for the moment they are also distinct. Since two copies of $A_i$ embed in the quotient torus $\mathbb{C}^*/\beta_i^{\mathbb{Z}}$ (one for the inside of the disk and one for the outside), we have

$$2\operatorname{mod}(A_i) \leq \frac{2\pi}{\log\beta_i} = \frac{2\pi}{L(C_i, f)}.$$

The combination of these inequalities yields:

$$\operatorname{mod}(p/q, X_\tau) \leq \pi \sum L(C_i, f)^{-1}.$$

This lower bound also holds when the cycles are not distinct; then we simply have more annuli $A_i$ embedded in a given torus $\mathbb{C}^*/\beta_j^{\mathbb{Z}}$.

For the upper bound, let $(\lambda_j)$ denote the multipliers of the repelling fixed points of $f^q$. Note that each cycle $C_i$ contributes $q$ fixed points, each with multiplier $\beta_i$. Combining Proposition 5.6 and Lemma 6.3, we obtain:

$$\begin{aligned}\frac{1}{q}\sum \frac{1}{\lambda_j - 1} &= \frac{1}{q}\sum{}' \frac{1}{\lambda_j - 1} + \sum \frac{1}{\beta_i - 1} = \frac{1-|\alpha^q|^2}{q|1-\alpha^q|^2} \\ &= \pi^{-1}\operatorname{mod}(p/q, X_\tau) + O(1).\end{aligned}$$

(Again, the prime indicates fixed points in $\bigcup C_i$ are excluded.) Since the cycles $C_i$ are compatible, there are no more than $d-1$ of them, and hence

$$\sum \frac{1}{\beta_i - 1} = \sum \left( \frac{1}{\log \beta_i} + O(1) \right) = \left( \sum L(C_i)^{-1} \right) + O(d).$$

This yields the upper bound in (6.1); and it also implies

$$\frac{1}{q}\sum{}' \frac{1}{\lambda_j - 1} = O(d).$$

That is, equation (6.2) holds for $r = q$.

To obtain (6.2) for other values of $r$, recall that by (5.2) we have $\operatorname{mod}(s/r, X_\tau) < 1$ whenever $s/r \neq p/q$. Thus if $q$ does not divide $r$, Lemma 6.3 implies

$$\frac{1}{r}\sum{}'(\lambda_j - 1)^{-1} \leq \frac{1-|\alpha^r|^2}{r|1-\alpha^r|^2} \leq \sup_s \operatorname{mod}(s/r, X_\tau) + O(1) = O(1);$$

while for $r = nq$ we obtain

$$\frac{1}{r}\sum{}' \frac{1}{\lambda_j - 1} + \frac{q}{r}\sum \frac{1}{\beta_i^n - 1} = \frac{1-|\alpha^r|^2}{r|1-\alpha^r|^2} = \frac{\operatorname{mod}(p/q, X_\tau)}{\pi n^2} + O(1),$$

which again implies (6.2), since (6.1) gives

$$\frac{q}{r}\sum \frac{1}{\beta_i^n - 1} = \frac{1}{n}\left( \sum \frac{1}{nL(C_i, f)} + O(1) \right) = \frac{\operatorname{mod}(p/q, X_\tau)}{\pi n^2} + O(d).$$

Finally note that equation (6.2) implies $L(C, f) > \epsilon_d \asymp 1/d > 0$, since any cycle $C$ of period $r$ and multiplier $\beta$, not among the $C_i$, contributes $1/(\beta - 1)$ to the sum $(1/r)\sum{}'(\lambda_j - 1)^{-1}$. ∎

# 7 Binding and renormalization

We conclude by proving the following compactness result.

**Theorem 7.1** *Let $(C_i)_1^n$ be a binding set of cycles of degree $d$. Then for any $M > 0$, the set of $f \in \mathcal{B}_d$ such that $\sum_i L(C_i, f) \le M$ has compact closure in* $\mathrm{MRat}_d$.

**Corollary 7.2** *The set of $f \in \mathcal{B}_d$ such that $\sum_i L(C_i, f) \le M$ and $|f'(0) - 1| \ge 1/M$ is compact.*

**Proof.** By Theorem 3.7, the only way a sequence $f_n$ can diverge in $\mathcal{B}_d$ but remain bounded in $\mathrm{MRat}_d$ is if $f_n'(0) \to 1$. ∎

**Definitions.** Sets $A, B \subset S^1$ are *unlinked* if they lie in disjoint connected sets; equivalently, if their convex hulls in the unit disk are disjoint. A map $f : X \to X$ with $X \subset S^1$ is *renormalizable* if there is a nontrivial partition of $X$ into disjoint, unlinked subsets $X_1, \ldots, X_n$, such that every $f(X_i)$ lies in some $X_j$.

We say a collection of degree $d$ cycles $C_1, \ldots, C_m$ is *binding* if $\deg(p_d | \bigcup C_i) = d$ and $p_d | \bigcup C_i$ is not renormalizable.

**Proof of Theorem 7.1.** Suppose to the contrary that we have a sequence $f_n \in \mathcal{B}_d$ with $\sum_i L(C_i, f_n) \le M$ that is divergent in moduli space. Let

$$D_n = \phi_{f_n}^{-1}\left(\bigcup C_i\right) \subset S^1$$

be the finite $f_n$-invariant set corresponding to the binding cycles. Since $f_n | S^1$ is expanding, we have $|f_n'| \le e^M$ on $D_n$.

Next we conjugate the entire picture by an affine transformation depending on $n$, so that $0 \in D_n$ and $\mathrm{diam}(D_n) = 1$. Then $S^1$ goes over to a circle $T_n \supset D_n$ invariant by $f_n$, and we still have $|f_n' | D_n| \le e^M$.

Pass to a subsequence such that $f_n \to (F, S) \in \overline{\mathrm{Rat}}_d$. Since $f_n$ diverges in $\mathrm{MRat}_d$, we have $\deg(F) < d$. Passing to a further subsequence, we can find a finite set $D$ containing zero and a circle $T \subset \widehat{\mathbb{C}}$ such that $D_n \to D$ and $T_n \to T$ in the Hausdorff topology. Note that $|D| > 1$ since $\mathrm{diam}\, D = 1$.

By Proposition 3.6, the map $f_n | D_n$ converges to $F | D$. But if $|D| = |\bigcup C_i|$, the map $F | (D \subset T)$ is combinatorially the same as $p_d | (\bigcup C_i \subset S^1)$, contradicting our assumption that $\deg(p_d | \bigcup C_i) = d$. Similarly, if $|D| < |\bigcup C_i|$, then the collapse of $D_n$ to $D$ provides an invariant partition for $\bigcup C_i$, contradicting our assumption that $p_d | \bigcup C_i$ is not renormalizable. ∎

**Examples.** The single cycle $C = (3, 6, 12, 24, 17)/31$ in degree 2 is already binding, as is any cycle of prime order with $\deg(p_d|C) = d$.

The first renormalizable cycle in degree 2 is $C = (1, 2, 4, 3)/5$. Although $\deg(p_2|C) = 2$, $L(C, f_n)$ remains bounded as $f_n \in \mathcal{B}_2$ diverges along the sequence specified by $f_n'(0) = -1 + 1/n$. Indeed, $f_n^2$ can be renormalized so that $C$ converges to the cycle of period 2 for $G(z) = z - 1/z$ [Ep]; and thus $L(C, f_n) \to \log 9$. For more details, see [Mc6, §14].

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